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Analytical properties of ultradiscrete Burgers equation and rule-184 cellular automaton

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Abstract. In this paper, we propose an ultradiscrete Burgers equation of which all the variables are discrete. The equation is derived from a discrete Burgers equation under an ultradiscrete limit and reduces to an ultradiscrete diffusion equation through the Cole–Hopf transformation. Moreover, it becomes a cellular automaton (CA) under appropriate conditions and is identical to rule-184 CA in a specific case. We show shock wave solutions and asymptotic behaviours of the CA exactly via the diffusion equation. Finally, we propose a particle model expressed by the CA and discuss a mean flux of particles.

1. Introduction

There are various degrees of discreteness of mathematical models used to describe physical phenomena, for example, the differential equation, difference equation, coupled map lattice and cellular automaton (CA) exist from the fully continuous model to the fully discrete one. Among them, the CA is the most discrete model of which variables are all discrete [1]. In particular, its dependent variable takes on a finite set of discrete values. Many CA have been proposed and used as simulators of phenomenon and analysed mathematically to grasp the behaviour of solutions. However, in the analysis, there often exists a difficulty peculiar to CA. For example, when we discuss linear stability or the asymptotic behaviour of difference equations, we often take a continuous limit of the equations. In the case of the CA, it is difficult to introduce such an approach owing to the discreteness of the dependent variable.

As a solution to the above problem, Tokihiro *et al* proposed a non-analytical limit named the 'ultradiscrete limit';

$$\lim_{\varepsilon\to+0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon} + \cdots) = \max(A, B, \ldots) \tag{1}$$

where $\max(A, B, \ldots)$ returns the maximum element in $\{A, B, \ldots\}$ [2]. They showed that the discrete Lotka–Volterra equation can reduce to the box and ball system under this limit. The former is a difference soliton equation with a continuous dependent variable [3]. The latter is a soliton CA defined by using boxes and balls [4]. Both have N-soliton solutions

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and those of the box and ball system are obtained exactly from those of the discrete Lotka–Volterra equation again by using the limit.

The ultradiscrete limit is not specific to soliton systems. Since equation (1) does not require integrability, it can be applied widely. Indeed, examples exist where the ultradiscrete limit is applied to a chaotic equation and to an elliptic function [5, 6].

In this paper, we apply the above ultradiscrete limit to a discrete analogue of the Burgers equation. Then, we obtain from the equation rule-184 CA of which rules are numbered following Wolfram [1]. The rule-184 CA (and its equivalence) is the only nonlinear CA preserving the number of 1's among the CA in the form

$$U_j^{t+1} = f(U_{j-1}^t, U_j^t, U_{j+1}^t) \tag{2}$$

where j is the site number, t is time, U is 0 or 1 and f is a Boolean function. Moreover, solutions to the rule-184 CA become steady at large enough t from any initial condition [7–9]. Owing to these remarkable properties, the CA is often used as a base of the traffic flow model [10, 11].

This paper is organized as follows. In section 2, using an ultradiscrete limit, we show that the discrete Burgers equation reduces to the ultradiscrete Burgers equation which can be a CA under a specific condition. We call this Burgers CA (BCA). The discrete Burgers equation reduces to a linear discrete diffusion equation and the BCA also reduces to an ultradiscrete diffusion equation. In section 3, we show that the BCA with a specific parameter becomes rule-184 CA. Moreover, we derive shock wave solutions to the BCA obtained from an ultradiscrete limit of discrete solutions. In section 4, we show an asymptotic behaviour of solutions to the BCA using an ultradiscrete diffusion equation. A solution from any initial state becomes steady at large enough time. In section 5, we propose a particle model expressed by the BCA. The mean flux of particles becomes constant at large enough time and the constant value only depends on the density of particles. In section 6, we give concluding remarks and future problems. Throughout the results, we use the properties of an ultradiscrete diffusion equation, as we do for the continuous Burgers equation.

2. Discrete Burgers equation and its ultradiscretization

First, we derive a discrete Burgers equation by using a discrete Cole–Hopf transformation. The continuous Burgers equation is

$$v_t = 2vv_x + v_{xx}. \tag{3}$$

It is well known that this equation can be linearized through the Cole–Hopf transformation given by

$$v = \frac{f_x}{f} \tag{4}$$

into the diffusion equation

$$f_t = f_{xx}. \tag{5}$$

To discretize equation (3), we utilize discrete analogues to equations (4) and (5) [12]. Discretizing both time and space variables in equation (5), a discrete diffusion equation

$$\frac{f_j^{t+1} - f_j^t}{\Delta t} = \frac{f_{j+1}^t - 2f_j^t + f_{j-1}^t}{(\Delta x)^2} \tag{6}$$

is obtained where Δt and Δx are lattice intervals in t and x respectively. Next we define a discrete analogue to the Cole–Hopf transformation

$$u_j^t \equiv c\frac{f_{j+1}^t}{f_j^t} \tag{7}$$

where c is a constant. Rewriting equation (6) with u_j^t in place of f_j^t we obtain

$$u_j^{t+1} = u_{j-1}^t \frac{1 + \frac{1-2\delta}{c\delta}u_j^t + \frac{1}{c^2}u_j^t u_{j+1}^t}{1 + \frac{1-2\delta}{c\delta}u_{j-1}^t + \frac{1}{c^2}u_{j-1}^t u_j^t} \tag{8}$$

where $\delta = \Delta t/(\Delta x)^2$. Assuming $v(j\Delta x, t\Delta t) = \frac{1}{\Delta x}\log\frac{u_j^t}{c}$ and taking the limits $\Delta x \to 0$ and $\Delta t \to 0$, we obtain equation (3) from equation (8). Therefore, we consider equation (8) to be a discrete analogue of the Burgers equation (3) and call equation (8) a 'discrete Burgers equation'.

Next, we 'ultradiscretize' equation (8), that is, discretize a dependent variable u using equation (1). Let us introduce a transformation of variables and parameters as follows

$$u_j^t = \mathrm{e}^{U_j^t/\varepsilon} \tag{9}$$

$$\frac{1-2\delta}{c\delta} = \mathrm{e}^{-M/\varepsilon} \tag{10}$$

$$c^2 = \mathrm{e}^{L/\varepsilon}. \tag{11}$$

Then, equation (8) reduces to

$$\begin{aligned} U_j^{t+1} = U_{j-1}^t &+ \varepsilon\log\left(1 + \exp\left(\frac{U_j^t - M}{\varepsilon}\right) + \exp\left(\frac{U_j^t + U_{j+1}^t - L}{\varepsilon}\right)\right) \\ &-\varepsilon\log\left(1 + \exp\left(\frac{U_{j-1}^t - M}{\varepsilon}\right) + \exp\left(\frac{U_{j-1}^t + U_j^t - L}{\varepsilon}\right)\right). \end{aligned} \tag{12}$$

Taking a limit $\varepsilon \to +0$ and using the relation (1), we obtain

$$U_j^{t+1} = U_{j-1}^t + \max(0, U_j^t - M, U_j^t + U_{j+1}^t - L) - \max(0, U_{j-1}^t - M, U_{j-1}^t + U_j^t - L). \tag{13}$$

Using identities

$$\max(A, B, \ldots) = -\min(-A, -B, \ldots) \tag{14}$$

$$\min(A, B, \ldots) + X = \min(A + X, B + X, \ldots) \tag{15}$$

the above equation becomes

$$U_j^{t+1} = U_j^t + \min(M, U_{j-1}^t, L - U_j^t) - \min(M, U_j^t, L - U_{j+1}^t). \tag{16}$$

If initial U and parameters M and L are all integer, then U for any t and j is always an integer. Thus, we obtain an equation with all discrete variables by the ultradiscrete limit (1). We call equation (16) the 'ultradiscrete Burgers equation'.

Under an appropriate condition, equation (16) becomes a CA. Assume that $M > 0$, $L > 0$ and $0 \leqslant U_j^t \leqslant L$ for any j at a certain t. Then, relations

$$\begin{aligned} &\min(M, U_{j-1}^t, L - U_j^t) \geqslant 0 \\ &\min(M, U_j^t, L - U_{j+1}^t) \geqslant 0 \\ &\min(M, U_{j-1}^t, L - U_j^t) + U_j^t = \min(M + U_j^t, U_{j-1}^t + U_j^t, L) \leqslant L \\ &\min(M, U_j^t, L - U_{j+1}^t) - U_j^t = \min(M - U_j^t, 0, L - U_{j+1}^t - U_j^t) \leqslant 0 \end{aligned} \tag{17}$$

hold. Therefore, $0 \leqslant U_j^{t+1} \leqslant L$ holds for any j. This means equation (16) under the above condition is equivalent to a CA with a value set $\{0, 1, \ldots, L\}$. We call this BCA.

Moreover, introducing a transformation

$$f_j^t = \exp(F_j^t/\varepsilon) \tag{18}$$

an ultradiscrete Cole–Hopf transformation

$$U_j^t = F_{j+1}^t - F_j^t + \frac{L}{2} \tag{19}$$

is obtained from equation (7) under the limit $\varepsilon \to +0$. Then, we obtain an ultradiscrete diffusion equation;

$$F_j^{t+1} = \max\left(F_{j-1}^t, F_j^t + \frac{L}{2} - M, F_{j+1}^t\right) \tag{20}$$

from equation (16). This equation can also be obtained from equation (6) with equation (18) under $\varepsilon \to +0$.

3. Relation to rule-184 CA and shock wave solutions of BCA

In this section, we put a restriction, $L \leqslant M$, on BCA for simplicity. Then, equation (16) reduces to

$$U_j^{t+1} = U_j^t + \min(U_{j-1}^t, L - U_j^t) - \min(U_j^t, L - U_{j+1}^t) \tag{21}$$

because any U_j^t satisfies $0 \leqslant U_j^t \leqslant L$ and is equal to or smaller than M.

Next, let us consider the case $L = 1$ for equation (21). The evolution rule for equation (21) is expressed symbolically by

$$\frac{U_{j-1}^t\, U_j^t\, U_{j+1}^t}{U_j^{t+1}} = \frac{000}{0}, \frac{001}{0}, \frac{010}{0}, \frac{011}{1}, \frac{100}{1}, \frac{101}{1}, \frac{110}{0}, \frac{111}{1}. \tag{22}$$

This rule is equivalent to that of rule-184 CA given by the following Boolean expression

$$U_j^{t+1} = (U_{j-1}^t \wedge \overline{U_j^t}) \vee (U_j^t \wedge U_{j+1}^t) \tag{23}$$

where $\wedge$, $\vee$ and $\overline{}$ denote AND, OR and NOT in Boolean operation respectively [1]. Therefore, we can conclude that BCA includes rule-184 CA as a special case. Note that various expressions using max and min functions can include the rule-184 CA. For example, by replacing $x \wedge y$, $x \vee y$ and $\overline{x}$ with $\min(x, y)$, $\max(x, y)$ and $1 - x$ respectively in equation (23), we obtain

$$U_j^{t+1} = \max(\min(U_{j-1}^t, 1 - U_j^t), \min(U_j^t, U_{j+1}^t)) \tag{24}$$

which is equivalent to rule-184 CA if U is restricted to 0 or 1. However, equations (21) and (24) are not equivalent if U can take an arbitrary integer value.

Then we derive solutions to equation (21) from shock wave solutions to the discrete Burgers equation (8). Let us assume that f_j^t has the following form;

$$f_j^t = 1 + \exp(kj + \omega t + \xi_0) \tag{25}$$

where k, ω and ξ_0 are constants. Substituting equation (25) into equation (6), we obtain a dispersion relation

$$\omega = \log(1 + \delta(\mathrm{e}^k - 2 + \mathrm{e}^{-k}). \tag{26}$$

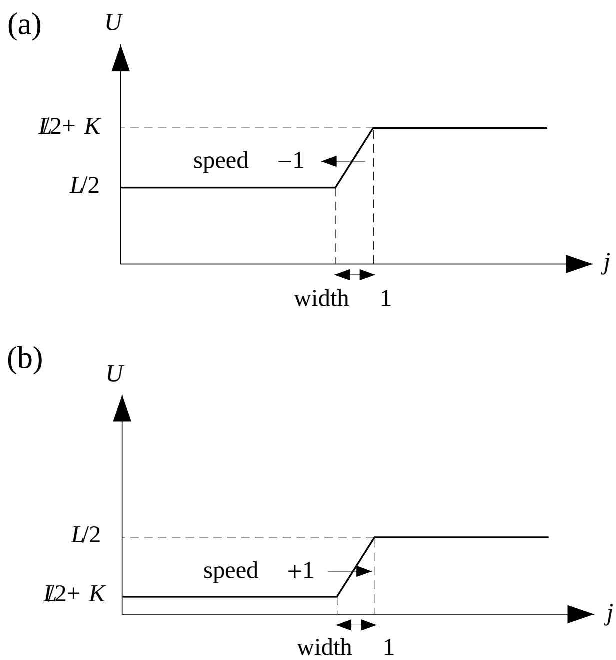


Figure 1. Shock wave solution. (a) $K > 0$, (b) $K < 0$.

Thus we obtain a solution

$$u_j^t = c\frac{f_{j+1}^t}{f_j^t} = c\frac{1+\exp(k(j+1)+\omega t+\xi_0)}{1+\exp(kj+\omega t+\xi_0)}. \tag{27}$$

This is a shock wave solution to the discrete Burgers equation (8). From this solution, we obtain a shock wave solution to the ultradiscrete Burgers equation (21) using the ultradiscrete limit. Assuming

$$k = \frac{K}{\varepsilon} \qquad \omega = \frac{\Omega}{\varepsilon} \qquad \xi_0 = \frac{\Xi_0}{\varepsilon} \tag{28}$$

and recalling equations (9) and (11), we obtain

$$U_j^t = \frac{L}{2} + \max(0, K(j+1)+\Omega t+\Xi_0) - \max(0, Kj+\Omega t+\Xi_0). \tag{29}$$

From equation (26) and the condition $L \leqslant M$, a dispersion relation

$$\Omega = |K| \tag{30}$$

is obtained. If $K > 0$, $\lim_{j\to-\infty} U_j^t = \frac{L}{2}$ and $\lim_{j\to+\infty} U_j^t = \frac{L}{2} + K$. If $K < 0$, $\lim_{j\to-\infty} U_j^t = \frac{L}{2} + K$ and $\lim_{j\to+\infty} U_j^t = \frac{L}{2}$. We can easily see that the above solution is a propagating wave with a speed -1 ($K > 0$) or $+1$ ($K < 0$) and its shape is like a step as shown in figure 1. Since any U_j^t must be an integer value from 0 to L, it is necessary for the above solution that L is an even positive integer, $|K| \leqslant L/2$ and Ξ_0 is an integer.

4. Asymptotic behavior of BCA

It is known that, in the case of the rule-184 CA (22) with a periodic boundary condition, U_j^t at large enough t becomes a steady solution [7–9]. There are two types of such solutions, one is $U_j^{t+1} = U_{j-1}^t$ and the other is $U_j^{t+1} = U_{j+1}^t$. Which type is selected depends on the total number of 1's. To date such behaviour has been mainly derived by pattern analysis on 1–0 sequences. In this paper, since we obtain the relation between rule-184 CA and the ultradiscrete Burgers equation reducible to the ultradiscrete diffusion equation, we can derive the asymptotic behaviour from analytic properties of the equations. Moreover, we can show the BCA as an extension of the rule-184 CA has similar properties to those described above.

First let us assume the space site of equation (16) is periodic with period K, that is, $U_j^t = U_{j+K}^t$. Then, we can easily see that $\sum_{i=1}^K U_i^t$ is constant for t. Therefore, defining ρ by

$$\rho = \frac{1}{KL}\sum_{i=1}^{K} U_i^t \tag{31}$$

ρ is constant. If we set the initial value U_j^0 at an initial time $t = 0$, we can construct F_j^0 from an inverse relation of equation (19),

$$F_j^0 = \begin{cases} \displaystyle\sum_{i=0}^{j-1}\left(U_i^0 - \frac{L}{2}\right) & \text{if } j \geqslant 1 \\ \displaystyle U_0^0 - \frac{L}{2} - \sum_{i=0}^{j}\left(U_i^0 - \frac{L}{2}\right) & \text{otherwise.} \end{cases} \tag{32}$$

Note that F_j^0 has a freedom of constant and we set $F_0^0 = 0$ in the above equation. Moreover, F_j^0 is not periodic and

$$F_{j+K}^0 - F_j^0 = \sum_{i=j}^{j+K-1}\left(U_i^0 - \frac{L}{2}\right) = KL(\rho - \tfrac{1}{2}). \tag{33}$$

Then, we can calculate F_j^t for $t > 0$ using equation (20) and obtain U_j^t by equation (19). This U_j^t also satisfies equation (16) with the above initial value U_j^0. That is, we can grasp the dynamics of BCA by equation (20) in place of equation (16).

Next we show the asymptotic behaviour of U_j^t at large enough t. We can assume K is even without loss of generality because we can consider the period is $2K$ if K is odd.

Case 1. $\frac{L}{2} \leqslant M$. From equation (20), we obtain

$$F_j^t = \max(\max(F_{j-t}^0, F_{j-t+2}^0, \ldots, F_{j+t}^0), \max(F_{j-t+1}^0, F_{j-t+3}^0, \ldots, F_{j+t-1}^0) + \alpha) \tag{34}$$

where $\alpha = \frac{L}{2} - M$.

Case 1.1. $\rho < \frac{1}{2}$. In this case, $F_{j+K}^0 < F_j^0$ holds from equation (33). Then

$$F_j^t = \max(\max(F_{j-t}^0, F_{j-t+2}^0, \ldots), \max(F_{j-t+1}^0, F_{j-t+3}^0, \ldots) + \alpha) \tag{35}$$

is derived for $t \geqslant \frac{K}{2}$. Therefore,

$$F_j^{t+1} = F_{j-1}^t \qquad \text{and} \qquad U_j^{t+1} = U_{j-1}^t \tag{36}$$

are obtained. Figure 2(a) shows an example of evolution.

By substituting equation (36) into equation (20), we obtain

$$0 = \max(0, F_j^t - F_{j-1}^t + \alpha, F_{j+1}^t - F_{j-1}^t) = \max(0, U_{j-1}^t - M, U_{j-1}^t + U_j^t - L). \tag{37}$$

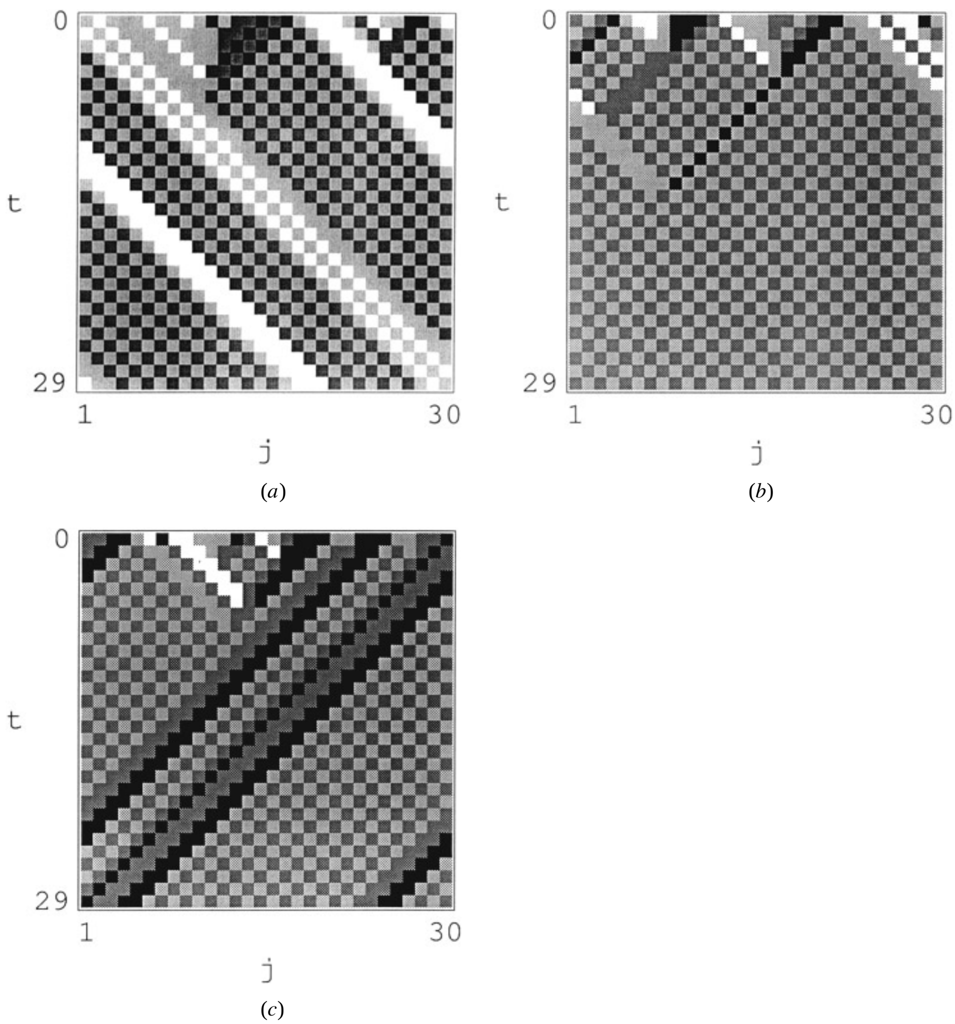


Figure 2. Time evolution from the random initial state for $L = 3$, $M = 2$ and $K = 30$. The black, dark grey, light grey and white squares denote sites with values 3, 2, 1, 0, respectively. (*a*) $\rho = 0.4$, (*b*) $\rho = 0.5$ and (*c*) $\rho = 0.6$.

From this condition,

$$U_j^t \leqslant M \qquad \text{and} \qquad U_j^t \leqslant L - U_{j+1}^t \tag{38}$$

hold for any j. In the case of rule-184 CA ($L = 1$, $M \geqslant 1$), the above condition means the sequence $U_1^t U_2^t \ldots U_K^t$ contains only 00, 01, 10 and not 11.

Case 1.2. $\rho = \frac{1}{2}$. Since $F_{j+K}^0 = F_j^0$,

$$F_j^t = \begin{cases} \max(\max(F_2^0, F_4^0, \ldots, F_K^0), \max(F_1^0, F_3^0, \ldots, F_{K-1}^0) + \alpha) & \text{if } j - t \text{ is even} \\ \max(\max(F_1^0, F_3^0, \ldots, F_{K-1}^0), \max(F_2^0, F_4^0, \ldots, F_K^0) + \alpha) & \text{otherwise} \end{cases} \tag{39}$$

is derived for $t \geqslant \frac{K}{2}$. Therefore, we obtain $F_j^{t+1} = F_{j\pm1}^t$ and $U_j^{t+1} = U_{j\pm1}^t$. Figure 2(*b*) shows an example of evolution. Substituting $F_j^{t+1} = F_{j\pm1}^t$ into equation (20), we find

$$U_j^t \leqslant M \qquad L - U_{j+1}^t \leqslant M \qquad \text{and} \qquad U_j^t = L - U_{j+1}^t \tag{40}$$

for any j. In the case of rule-184 CA, the above condition means the sequence $U_1^t U_2^t \ldots U_K^t$ is $0101\ldots01$ or $1010\ldots10$.

Case 1.3. $\rho > \frac{1}{2}$. By using a similar discussion to case 1.1, we obtain

$$F_j^{t+1} = F_{j+1}^t \qquad \text{and} \qquad U_j^{t+1} = U_{j+1}^t \tag{41}$$

and

$$L - U_{j+1}^t \leqslant U_j^t \qquad \text{and} \qquad L - U_j^t \leqslant M \tag{42}$$

for $t \geqslant \frac{K}{2}$. Figure 2(*c*) shows an example of evolution. In the case of rule-184 CA, the above condition means the sequence $U_1^t U_2^t \ldots U_K^t$ contains only 01, 10, 11 and not 00.

Case 2. $\frac{L}{2} > M$. From equation (20), we obtain

$$F_j^t = \max(F_{j-t}^0, F_{j-t+1}^0 + \alpha, \ldots, F_j^0 + t\alpha, \ldots, F_{j+t-1}^0 + \alpha, F_{j+t}^0). \tag{43}$$

Case 2.1. $\rho < \frac{M}{L}$. In this case, $F_{j+K}^t + K\alpha < F_j^t$ holds. Therefore,

$$F_j^t = \max(F_{j-t}^0, F_{j-t+1}^0 + \alpha, \ldots, F_{j-t+K-1}^0 + (K-1)\alpha) \tag{44}$$

is derived for $t \geqslant K$. Then, we obtain

$$F_j^{t+1} = F_{j-1}^t \qquad \text{and} \qquad U_j^{t+1} = U_{j-1}^t \tag{45}$$

and

$$U_j^t \leqslant M \qquad \text{and} \qquad U_j^t \leqslant L - U_{j+1}^t. \tag{46}$$

Figure 3(*a*) shows an example of evolution.

Case 2.2. $\frac{M}{L} \leqslant \rho \leqslant 1 - \frac{M}{L}$. In this case, since $|F_{j+K}^t - F_j^t| = KL|\rho - \frac{1}{2}| \leqslant K\alpha$, we can derive

$$F_{j\pm K}^t \leqslant F_j^t + K\alpha. \tag{47}$$

Therefore, we obtain

$$\begin{aligned} F_j^t = \max(&F_{j-K+1}^0, F_{j-K+2}^0 + \alpha, \ldots, F_j^0 + (K-1)\alpha, \ldots, \\ &F_{j+K-2}^0 + \alpha, F_{j+K-1}^0) + (t - K + 1)\alpha \end{aligned} \tag{48}$$

from equation (43) for $t \geqslant K$. Using this relation,

$$F_j^{t+1} = F_j^t + \alpha \qquad \text{and} \qquad U_j^{t+1} = U_j^t \tag{49}$$

and

$$M \leqslant U_j^t \leqslant L - M \tag{50}$$

hold for $t \geqslant K$. Figure 3(*b*) shows an example of evolution.

Case 2.3. $\rho > 1 - \frac{M}{L}$. By using a similar discussion to case 2.1,

$$F_j^{t+1} = F_{j+1}^t \qquad \text{and} \qquad U_j^{t+1} = U_{j+1}^t \tag{51}$$

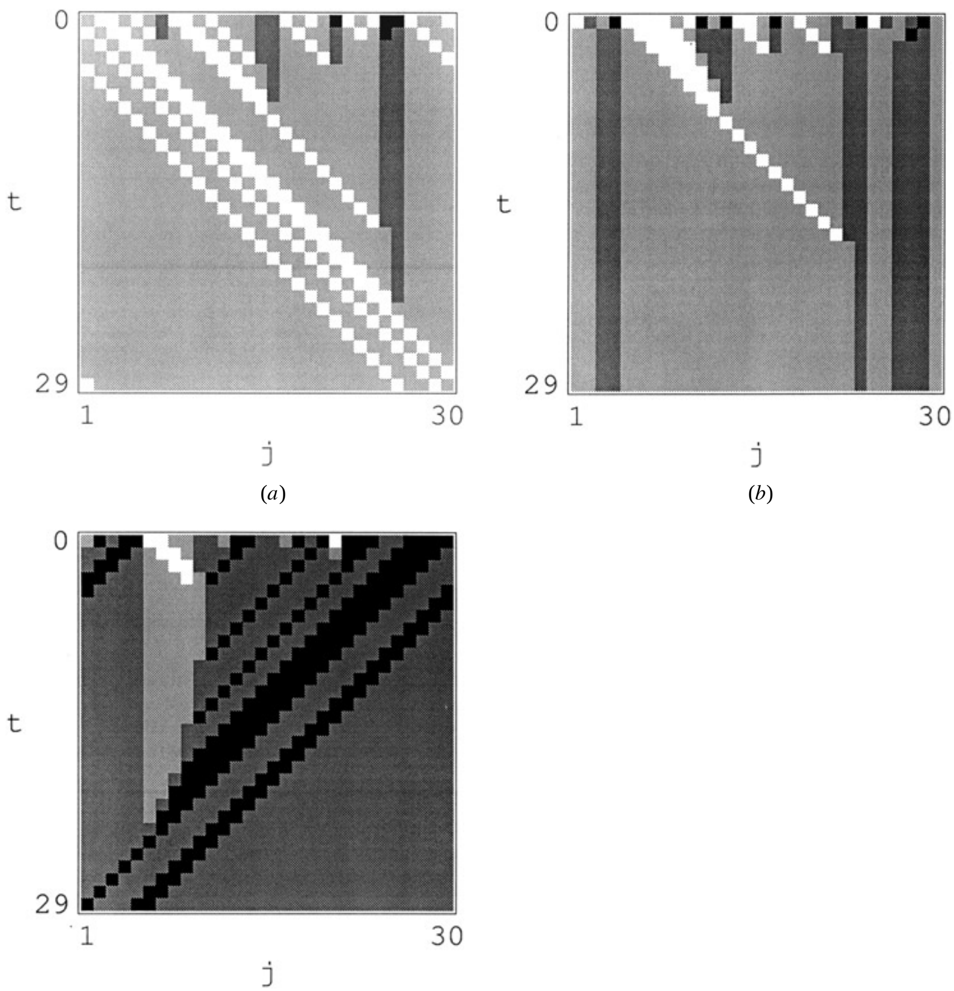
and

$$L - U_{j+1}^t \leqslant U_j^t \qquad \text{and} \qquad L - U_j^t \leqslant M \tag{52}$$

hold for $t \geqslant K$. Figure 3(*c*) shows an example of evolution.

5. Particle model expressed by BCA

In rule-184 CA (22), the number of 1's is conserved for time and the evolution rule is interpreted as the following motion of particles [9].



(*a*) (*b*) (*c*)

Figure 3. Time evolution from the random initial state for $L = 3$, $M = 1$ and $K = 30$. Black, dark grey, light grey and white squares denote sites with values 3, 2, 1, 0, respectively. (*a*) $\rho = 0.3$, (*b*) $\rho = 0.4$ and (*c*) $\rho = 0.7$.

• Each site can hold one particle at the most. U_j^t denotes the number of particles at site j and time t. From t to $t+1$, particles move to their right site if the site is empty at t and do not move otherwise.

BCA (16) including rule-184 CA as a special case can express the following particle model.

• Each site can hold L particles at most. U_j^t denotes the number of particles at site j and time t. From t to $t+1$, particles at site j can move to site $j+1$. The maximum number of movable particles is M. Under this restriction, they move to the vacant space at site $j+1$ as many as they can.

According to the above rule, the number of moveable particles at site j and time t is $\min(M, U_j^t, L-U_{j+1}^t)$. Therefore, U_j^{t+1} is calculated by equation (16). We can easily see from the above rule that the total number of particles is conserved.

Next let us consider a mean flux of particles [11]. If q^t denotes the mean flux, it is defined by

$$q^t = \frac{1}{KL}\sum_{i=1}^{K}\min(M, U_i^t, L-U_{i+1}^t). \tag{53}$$

From the results of the previous section, we can show that q^t becomes constant at large enough t and the constant value depends only on the particle density ρ and not on the initial distribution of particles. For example, in case 1.1 ($\frac{L}{2} \leqslant M,\ \rho < \frac{1}{2}$), we find

$$q^t = \frac{1}{KL}\sum_{i=1}^{K}U_i^t = \rho \qquad \left(t \geqslant \frac{K}{2}\right) \tag{54}$$

since $\min(M, U_j^t, L-U_{j+1}^t) = U_j^t$ from equation (38). By using similar discussions, in the case of $\frac{L}{2} \leqslant M$,

$$q^t = \begin{cases} \rho & \text{if } \rho \leqslant \frac{1}{2} \\ 1-\rho & \text{otherwise} \end{cases} \qquad \left(t \geqslant \frac{K}{2}\right) \tag{55}$$

and in the case of $\frac{L}{2} > M$,

$$q^t = \begin{cases} \rho & \text{if } \rho < \dfrac{M}{L} \\ \dfrac{M}{L} & \text{if } \dfrac{M}{L} \leqslant \rho \leqslant 1-\dfrac{M}{L} \\ 1-\rho & \text{otherwise} \end{cases} \qquad (t \geqslant K). \tag{56}$$

In particular, we can show that q^t increases monotonically on t in the case of $\frac{L}{2} \leqslant M$. Using equation (16), $\sum_{i=1}^{K} U_i^t =$ constant and $\frac{L}{2} \leqslant M$, we obtain

$$\begin{aligned} q^{t+1} - q^t &= \frac{1}{KL}\sum_{i=1}^{K}\{\min(M, U_i^{t+1}, L-U_{i+1}^{t+1}) - \min(M, U_i^t, L-U_{i+1}^t)\} \\ &= \frac{1}{KL}\sum_{i=1}^{K}\max(0, g(U_i^t, U_{i+1}^t, U_{i+2}^t, U_{i+3}^t)) \geqslant 0 \end{aligned} \tag{57}$$

where

$$\begin{aligned} g(a_0, a_1, a_2, a_3) = \min(&2L, L+M+a_3, L+a_2+a_3, \\ &M+a_1+a_2+a_3, a_0+a_1+a_2+a_3) \\ &-\min(L+M+a_1, L+a_0+a_1, 2M+a_1+a_3, M+a_0+a_1+a_3). \end{aligned}$$

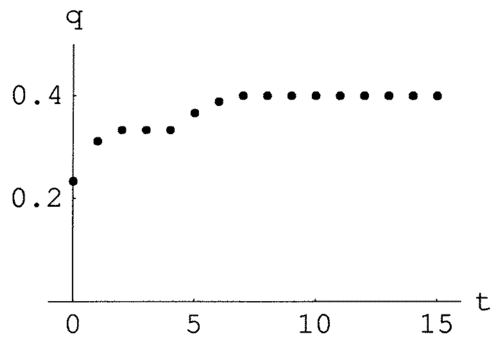


Figure 4. Monotonical increase of q^t. The same data as in figure 2(*c*) are used.

Figure 4 shows an evolution of q^t obtained from the same data as in figure 2(*c*). Since q^t is a finite value, q^t becomes constant at $t \gg 0$. Then $g(U_j^t, U_{j+1}^t, U_{j+2}^t, U_{j+3}^t) = 0$ is obtained for any j and we can derive the same results as in the previous section concerning the asymptotic behaviour.

6. Concluding discussions

In this paper, the main results are as follows.

(i) The relation between the Burgers equation and rule-184 CA is clarified via discrete and ultradiscrete Burgers equations. Under specific conditions, ultradiscrete Burgers equation can be BCA including rule-184 CA.

(ii) Shock wave solutions exist in BCA which is derived from discrete shock wave solutions under an ultradiscrete limit.

(iii) Any solution to BCA with periodic boundary conditions becomes steady at large enough time. The sequence of U_j^t converges to a stable pattern shifting to the right or left, or to a static pattern. Only ρ decides which pattern is selected.

(iv) BCA expresses an evolutional system of moving particles. The mean flux of particles becomes constant at large enough time. The constant value depends only on the density of particles. In the specific case, the mean flux increases monotonically on time.

In the above results, the linear diffusion equation obtained using the Cole–Hopf transformation plays an important role. In the discrete Burgers equation, shock wave solutions and asymptotic behaviour can be grasped through the diffusion equation. In the BCA, corresponding results are obtained by parallel discussions. We can consider such a relation between the discrete equation and CA can introduce a new viewpoint to discrete analysis.

On the other hand, BCA and rule-184 CA are easy to analyse since they are related to the discrete Burgers equation which can be analysed exactly. CA exist in the form of equation (2) of which solutions show chaotic behaviour. If we discuss such types of CA, it may be difficult to show what structure of CA is preserved in the corresponding discrete equation. This problem is left for future discussion.

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